A New Method for Designing Wide-Band Parametric Amplifiers*

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Summary—**A practical method of designing wide-band parametric amplifiers operated with a circulator is described. For amplifiers with an initially series-tuned varactor, it is possible to find simple relations between Butterworth and Chebyshev responses of low-pass filters and desired gain responses of maximally flat or equal-ripple type. These relations are shown to hold for most practical varactors. For amplifiers with an initially series-tuned varactor, simple expressions for the limiting gain bandwidth product are given. It is also shown how filters should be chosen to give stable amplifiers.**

I. Introduction

Since the wide-band potentialities of single-diode amplifiers using filter circuits were first discovered by Seidel and Herrman,[1] various methods for designing proper filter circuits have been treated in the literature.

Seidel and Herrman[1] give design criteria for a filter circuit of a degenerate amplifier, using the approach of setting derivatives of the gain function equal to zero at midband.

Matthaei[2] gives expressions fort he gain, suitable for wide-band design, using the complete equivalent circuit of the varactor, and shows that considerable bandwidths can be obtained, using properly dimensioned band-pass filters in the signal and idler circuits. However, no direct way of choosing the proper filters is given and a certain amount of cut and try is involved.

Kuh and Fukada[3] treat a parallel-tuned ideal varactor and give the relations between the power gain and the reflection coefficient of the lossless coupling networks terminated by certain resistances. Starting from Bode's theorem on reflection coefficient limitation, simple equations for the limitation on gain and bandwidth are derived. Butterworth filters are shown to give maximally flat gain.

This paper is based on the results of the mentioned papers, but the results are extended to give relations between a desired Butterworth or Chebyshev gain response and the Butterworth of Chebyshev filters giving the desired gain response. These relations are given for ideal and practical varactors and simplify the design procedure for wide-band parametric amplifiers. Using these relations, it is also possible to study how the limitation on gain and bandwidths is affected by the loss resistance and series inductance in the varactor. It is also shown how the filters should be chosen to give stable amplifiers.

II. Formulation of Butterworth or Chebyshev Matching

To be able to determine proper filters giving wide-band parametric amplifiers, different ways of formulating Butterworth or Chebyshev matching, with a lossless filter between positive resistances, must be studied. The low-pass equivalent is chosen as shown in Fig. 1.

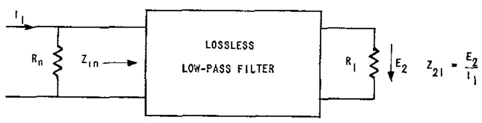

Fig. 1—Circuit used for expressing matching conditions.

As given by Weinberg,[4,5] the Butterworth and Chebyshev matching can be expressed by imposing some restriction on the transfer impedance Z_{21}. For a Butterworth response we have

$$|Z_{21}|^2 = \frac{T}{1+\omega^{2n}}. \tag{1}$$

For a Chebyshev response we have

$$|Z_{21}|^2 = \frac{T}{1+\epsilon^2 V_n^2(\omega)}, \qquad V_n(\omega) = \cos(n \cos^{-1}\omega). \tag{2}$$

By expressing the denominator in Z_{21} in Butterworth or Chebyshev polynomials the matching filter can be determined.

The restrictions on the transfer impedance lead to some restrictions on the reflection coefficient. We have

$$P_{\text{absorbed}} = \frac{1}{R_1}|E_2|^2 \quad P_{\text{available}} = \tfrac{1}{4}R_n|I_1|^2,$$

$$1 - |\rho|^2 = \frac{P_{\text{absorbed}}}{P_{\text{available}}}. \tag{3}$$

* Received June 22, 1962; revised manuscript received October 2, 1962. This work was carried out at Stanford University, Stanford, Calif., and supported by the U. S. Air Force under contract 33(616)-7944.

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1 H. Seidel and G. F. Herrmann, "Circuit aspects of parametric amplifiers," 1959 IRE WESCON Convention Record, pt. 2. pp. 83–90.

2 G. L. Matthaei, "A study of the optimum design of wide-band parametric amplifiers and up-converters," IRE Trans. on Microwave Theory and Techniques, vol. MTT-9, pp. 23–38; January, 1961.

3 E. S. Kuh and M. Fukada, "Optimum synthesis of wide-band parametric amplifiers and up-converters," IRE Trans. on Circuit Theory, vol. CT-9, pp. 410–415; December, 1961.

4 L. Weinberg, "ABCD-network design, easy as pie," *Proc. National Electronics Conf.*, Chicago, Ill., October 7–9, 1957, vol. 13, pp. 1057–1066; 1957.

5 L. Weinberg, "Network design by use of modern synthesis techniques and tables," *Proc. National Electronics Conf.*, Chicago, Ill., October 1–3, 1956, vol. 12, pp. 794–817; 1956.

which give for the respective Butterworth and Chebyshev responses,

$$\begin{cases} |\rho|^2 = \dfrac{\rho_0^2 + \omega^{2n}}{1+\omega^{2n}} \\ |\rho|^2 = \dfrac{\rho_0^2 + \epsilon^2 V_n^2(\omega)}{1+\epsilon^2 V_n^2(\omega)} \\ \rho_0^2 = 1 - \dfrac{4T}{R_1 R_n} = \left(\dfrac{R_1 - R_n}{R_1 + R_n}\right)^2. \end{cases} \tag{4}$$

The restrictions on the reflection coefficient ρ can also be expressed as restrictions on the input impedance $Z_{in} = R_{in} + jX_{in}$ in the complex impedance plane. We can write the reflection coefficient at the input as

$$\rho = \frac{Z_{in} - R_n}{Z_{in} + R_n}. \tag{5}$$

Then the Butterworth restriction on $|\rho|^2$ can be rewritten as

$$Z_{in} = R_n \frac{1 + \rho_0^2 + 2\omega^{2n}}{1 - \rho_0^2} + R_n \frac{2\rho_0 \sqrt{1 + \dfrac{1+\rho_0^2}{\rho_0^2}\omega^{2n} + \dfrac{1}{\rho_0^2}\omega^{4n}}}{1 - \rho_0^2} e^{j\phi}. \tag{6}$$

Thus the $Z_{in}(j\omega)$ curve in the complex impedance plane is a circle with frequency dependant center and radius. As the frequency derivatives $d^m/d\omega^m$ of the center and radius are zero for $m < 2n$, increasing n means that higher and higher derivatives of R_{in} and X_{in} have values that make Z_{in} follow a constant ρ-circle in the complex impedance plane.

$$Z_{in} = R_n \frac{1+\rho_0^2}{1-\rho_0^2} + R_n \frac{2\rho_0}{\rho - \rho_0^2} e^{j\phi} \tag{7}$$

The Chebyshev restriction on $|\rho|^2$ can be rewritten as

$$Z_{in} = R_n \frac{1 + \rho_0^2 + 2\epsilon^2 V_n^2(\omega)}{1 - \rho_0^2} + R_n \frac{2\rho_0 \sqrt{1 + \dfrac{1+\rho_0^2}{\rho_0^2}\epsilon^2 V_n^2(\omega) + \dfrac{1}{\rho_0^2}\epsilon^4 V_n^4(\omega)}}{1 - \rho_0^2} e^{j\phi}. \tag{8}$$

Thus the $Z_{in}(j\omega)$ curve in the complex impedance plane shall approximate a ρ circle, where ρ oscillates between ρ_0 and $\sqrt{\rho_0^2 + \epsilon^2 / 1 + \epsilon^2}$. As n is increased higher and higher derivatives of R_{in} and X_{in} have values that make Z_{in} follow a "Chebyshev circle." In Fig. 2 we show the Z_{in} curves for ideal Butterworth and Chebyshev response.

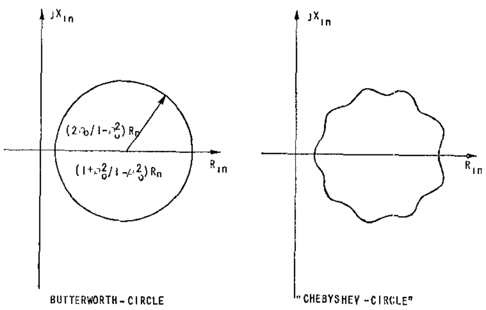

Fig. 2—Z_{in} curves for ideal Butterworth and Chebyshev response.

If R_{in} and X_{in} are expanded in Taylor series,

$$Z_{in} = \sum_{n=0}^{\infty} \frac{1}{n!} R_{in}^{(n)} \omega^n + j \sum_{n=0}^{\infty} \frac{1}{n!} X_{in}^{(n)} \omega^n$$

$$R_{in}^{(n)} = \frac{d^n R_{in}}{d\omega^n}(\omega = 0) \qquad X_{in}^{(n)} = \frac{d^n X_{in}}{d\omega^n}(\omega = 0). \tag{9}$$

Butterworth responses of increasing order can be expressed as relations between the derivatives.

$$n = 2 \begin{cases} n = 1 \begin{cases} \left[R_{in}^{(0)} - \dfrac{1+\rho_0^2}{1-\rho_0^2} R_n\right]^2 + [X_{in}^{(0)}]^2 = \dfrac{4\rho_0^2}{(1-\rho_0^2)^2} R_n^2 \\ \left[R_{in}^{(0)} - \dfrac{1+\rho_0^2}{1-\rho_0^2} R_n\right] R_{in}^{(1)} + X_{in}^{(0)} X_{in}^{(1)} = 0 \end{cases} \\ \left[R_{in}^{(0)} - \dfrac{1+\rho_0^2}{1-\rho_0^2} R_n\right] R_{in}^{(2)} + [R_{in}^{(1)}]^2 + X_{in}^{(0)} X_{in}^{(2)} + [X_{in}^{(1)}]^2 = 0 \\ \dfrac{1}{3}\left[R_{in}^{(0)} - \dfrac{1+\rho_0^2}{1-\rho_0^2} R_n\right] R_{in}^{(3)} + R_{in}^{(1)} R_{in}^{(2)} + \dfrac{1}{3} X_{in}^{(0)} X_{in}^{(3)} + X_{in}^{(1)} X_{in}^{(2)} = 0. \end{cases} \tag{10}$$

Similar relations can be found for the Chebyshev response.

These relations give the ordinary matching low-pass filter listed in filter tables.[5] The relations giving corresponding band-pass filters are found if Taylor expansions around a chosen center frequency ω_0 are used. The given relations are useful when designing a matching filter between a source impedance R_n and a frequency dependant resistance $R_1(\omega)$. Similar problems arise when designing filters for wide-band parametric amplifiers and tuned diode amplifiers. The filter elements can be determined by using (10), and the general form of the band-pass filter will be a filter, with series- and parallel-resonances detuned from the center frequency ω_0, as shown in Fig. 3.

If the phase of the reflection coefficient must be controlled at ω_0, this condition must be added to (10) and a compensating reactance must be inserted on one filter end, in series, if the last resonance is in parallel and vice versa.

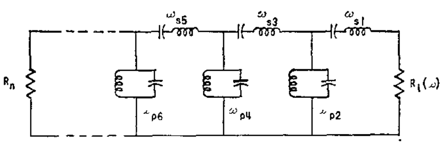

Fig. 3—Band-pass filter matching into a frequency-dependent resistance.

III. Signal-idler Interaction

The frequency mixing in variable-reactance elements has been treated extensively in the literature.[6,7] We assume that we have a varactor with a capacitive element that varies linearly with voltage

$$Q = C_0V_d + C_1V_d^2 \qquad (11)$$

in which

V_d = voltage over the capacitance,
V_0 = dc voltage over the capacitance,
C_s = static dc capacitance $= C_0 + C_1V_0$,
C_d = dynamic ac capacitance $= C_0 + 2C_1V_0$.

Furthermore we assume that all unwanted sidebands are shortcircuited over the capacitance, so that the ac voltage over the capacitance can be written

$$\begin{gathered}\tilde{V}_d = \overline{V}_{pd}e^{j\omega_pt} + \overline{V}_{sd}e^{j\omega_st} + \overline{V}_{id}e^{j\omega_it} \\ \omega_s + \omega_i = \omega_p,\end{gathered} \qquad (12)$$

when

$\overline{V}_{pd}$ = pump voltage,
$\overline{V}_{sd}$ = signal voltage,
$\overline{V}_{id}$ = idler voltage.

If we put the applied voltages into (11), and assume that the pump voltage is large compared to the signal and idler voltages, we obtain the following relations between signal and idler voltages and currents in the capacitive element,

$$\begin{cases}\overline{I}_{sd} = j\omega_sC_d\overline{V}_{sd} + j\omega_sC_1\overline{V}_{pd}\overline{V}_{id}^* \\ \overline{I}_{id} = j\omega_iC_d\overline{V}_{id} + j\omega_iC_1\overline{V}_{pd}\overline{V}_{sd}^*,\end{cases} \qquad (13)$$

or inverted,

$$\begin{cases}\overline{V}_{sd} = -j\dfrac{1}{\omega_s(1-|\bar{\alpha}|^2)C_d}\overline{I}_{sd} - j\dfrac{\bar{\alpha}}{\omega_i(1-|\bar{\alpha}|^2)C_d}\overline{I}_{id}^* \\ \overline{V}_{id} = -j\dfrac{1}{\omega_i(1-|\bar{\alpha}|^2)C_d}\overline{I}_{id} - j\dfrac{\bar{\alpha}}{\omega_s(1-|\bar{\alpha}|^2)C_d}\overline{I}_{sd}^*\end{cases}$$

$$\bar{\alpha} = \frac{C_1\overline{V}_{pd}}{C_d}. \qquad (14)$$

The varying capacitance gives a coupling between signal and idler circuits, which can be represented as a negative resistance. The coupling equations (13) and (14) can be used for two different representations of the signal-idler interaction, given in Fig. 4. As we assume that the parametric amplifier is operated with an ideal circulator, the amplification is equal to the reflection coefficient in the signal circuit.

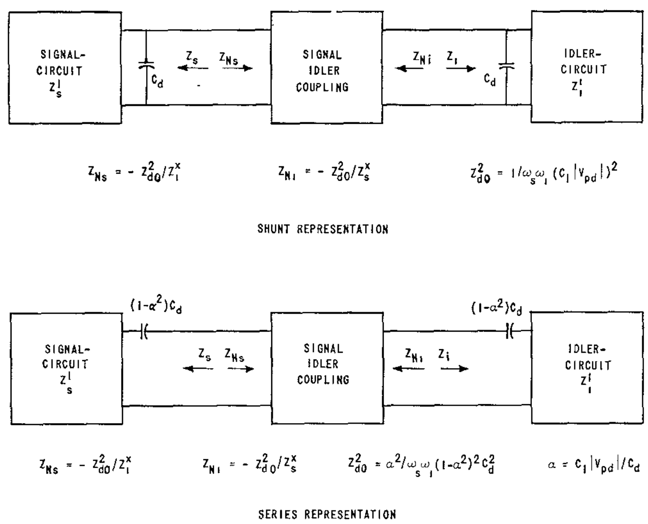

Fig. 4—Representation of signal-idler interaction.

IV. Wide-band Design for Ideal Varactor

To start with, we consider an ideal veractor where we can neglect the loss resistance and series inductance. Using either shunt or series representation with given notations the voltage gain G can be written

$$G = \frac{Z_{Ns} - Z_s^*}{Z_{Ns} + Z_s} = \frac{Z_{d0}^2 + Z_s^*Z_i^*}{Z_{d0}^2 - Z_sZ_i^*}. \qquad (15)$$

It is assumed that Z_s and Z_i are built up from band-pass filters with resistive terminations and so dimensioned that Z_s and Z_i are symmetrical around the center frequencies ω_{s0} and ω_{i0}

$$Z_i^* = K^2Z_s. \qquad (16)$$

[6] H. Heffner and G. Wade, "Gain, bandwidth, and noise characteristics of the variable-parameter amplifier," *J. Appl. Phys.*, vol. 29, pp. 1321–1331; September, 1958.
[7] H. E. Rowe, "Some general properties of nonlinear elements. II. Small signal theory," Proc. IRE, vol. 46, pp. 850–860; May, 1958.

The constant K is later chosen to give the signal and idler filters the same bandwidth capabilities which are necessary to obtain optimum bandwidth. The gain now can be written

$$G = \frac{(Z_{d0}/K)^2 + Z_s Z_s^*}{(Z_{d0}/K)^2 - Z_s^2} = \frac{(KZ_{d0})^2 + Z_i Z_i^*}{(KZ_{d0})^2 - Z_i^{*2}} \cdot \qquad (17)$$

Using the results of Kuh and Fukada[3] we can define a reflection coefficient ρ

$$\rho = \frac{Z_s - Z_{d0}/K}{Z_s + Z_{d0}/K} = \frac{Z_i^* - KZ_{d0}}{Z_i^* + KZ_{d0}}, \qquad (18)$$

and then the gain can be written as

$$|G|^2 = \frac{(1 + |\rho|^2)^2}{4|\rho|^2} \cdot \qquad (19)$$

The interpretation of (19) is, that to obtain a maximally flat gain response, the signal and idler circuits consist of Butterworth band-pass filters matching the signal and idler terminations into Z_{d0}/K and KZ_{d0}. To obtain a equal ripple gain response the signal and idler circuits consists of Chebyshev band-pass filters matching the signal and idler terminations into Z_{d0}/K and KZ_{d0} with a ripple determined by the gain ripple.

When designing a maximally flat gain response we put a Butterworth restriction on the reflection coefficient given in (18)

$$|\rho|^2 = \frac{\rho_0^2 + \omega^{2n}}{1 + \omega^{2n}}, \qquad (20)$$

which inserted in (19) gives

$$|G|^2 = \frac{(\rho_0^2 + 1)^2 + 4(\rho_0^2 + 1)\omega^{2n} + 4\omega^{4n}}{4\rho_0^2 + 4(\rho_0^2 + 1)\omega^{2n} + 4\omega^{4n}}$$

$$\approx \frac{G_0^2 + f(\rho_0)\omega^{2n}}{1 + f(\rho_0)\omega^{2n}}, \qquad (21)$$

where

$$\rho_0 = G_0 - \sqrt{G_0^2 - 1} \qquad (22)$$

When designing an equal ripple gain response we put a Chebyshev restriction on the reflection coefficient given in (18)

$$|\rho|^2 = \frac{\rho_0^2 + \epsilon_p^2 V_n^2(\omega)}{1 + \epsilon_p^2 V_n^2(\omega)}, \qquad (23)$$

which inserted in (19) gives

$$|G|^2 \approx \frac{G_0^2 + \epsilon_g^2 V_n^2(\omega)}{1 + \epsilon_g^2 V_n^2(\omega)} \qquad (24)$$

where

$$\begin{cases} \rho_0 = G_0 - \sqrt{G_0^2 - 1} \\ \epsilon_p^2 = \dfrac{(G_0 - \sqrt{G_0^2 - 1})\epsilon_g^2 + G_0 - \sqrt{G_0^2 + \epsilon_g^2}}{\sqrt{G_0^2 + \epsilon_g^2} - \sqrt{G_0^2 - 1}} \end{cases} \cdot \qquad (25)$$

We now assume that Z_{d0} is constant over the band of interest ($\omega_s \omega_i \approx \omega_{s0} \omega_{i0}$). It is now possible to design the signal and idler filters for a desired gain response by taking the proper low-pass filter with a response given in (20) or (23) and transform them into their band-pass equivalent with incorporation of the diode capacitance in the filter structure. If the varactor is parallel tuned, the shunt representation of the signal-idler iteraction is used and if series tuned, the series representation. When using this procedure it must be assumed that the signal and idler circuits are independent of each other.

V. Wide-Band Design for Practical Varactor

When determining the filter for a practical varactor, the design given in IV is not adequate. A practical varactor contains a series loss resistance R and a series inductance L_s shown in Fig. 5. For the given equivalent circuit it is convenient to use the series representation of the signal-idler interaction given in Fig. 6.

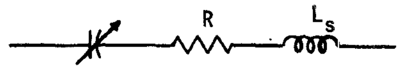

Fig. 5—Equivalent circuit for a practical varactor.

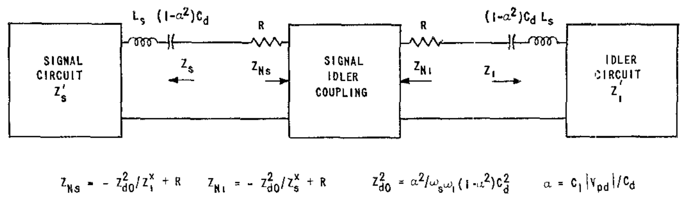

Fig. 6—Representation of signal-idler interaction for a practical varactor.

As we assumed that the parametric amplifier is operated with an ideal circulator the voltage gain is equal to the reflection coefficient in the signal circuit.

With the given notations the voltage gain G can be written

$$G = \frac{Z_{Ns} + R - Z_s^*}{Z_{Ns} + R + Z_s} = \frac{Z_{d0}^2 + (Z_s^* - R)(Z_i^* + R)}{Z_{d0}^2 - (Z_s + R)(Z_i^* + R)} \cdot \qquad (26)$$

As before we assume that Z_s and Z_i are built up so that (16) holds.

A) Degenerate Amplifier

To start with, we restrict the treatment to a degenerate amplifier where we have $K^2=1$. The single sideband voltage gain G for a degenerate amplifier now can be written

$$G = \frac{Z_{d0}^2 + (Z_s^* - R)(Z_s + R)}{Z_{d0}^2 - (Z_s + R)^2}. \tag{27}$$

It is practical to start with an investigation of the Z_s curve that gives a constant voltage gain G_0. This Z_s curve intersects the real R_s axis at two points.

$$\begin{cases} R_{s1} = \sqrt{\dfrac{G_0 - 1}{G_0 + 1}}\, Z_d \\ R_{s2} = \sqrt{\dfrac{G_0 + 1}{G_0 - 1}}\, Z_d \\ Z_d = \sqrt{Z_{d0}^2 + \dfrac{R^2}{G_0^2 - 1}} - R\dfrac{G_0}{\sqrt{G_0^2 - 1}}, \end{cases} \tag{28}$$

if a circle is drawn through these points we have

$$\begin{cases} Z_s = Z_i^* = \dfrac{1 + \rho_0^2}{1 - \rho_0^2} Z_d + \dfrac{2\rho_0}{1 - \rho_0^2} Z_d e^{j\phi} \\ \rho_0 = G_0 - \sqrt{G_0^2 - 1} \end{cases} \tag{29}$$

which is the Z_s curve giving a constant reflection coefficient ρ_0 into Z_d. This curve can be used as an approximation of the Z_s curve giving a constant voltage gain G_0 and to investigate the accuracy of the approximation we calculate the gain obtained by using the approximation for different values of G_0 and R/Z_{d0}. The results are given in Fig. 7 as $\Delta G[\text{db}] = 10 \log |G_{\max}|^2 - 10 \log G_0^2$ as a function of R/Z_{d0} with G_0 as a parameter.

The calculations show that if a gain fluctuation of 0.4 db is tolerated the approximation is valid for $R/Z_{d0} < 0.9$. This is not a severe restriction as the varactor is too lossy to give amplification when $R/Z_{d0} > 1.0$. For values of R/Z_{d0} where the approximation is accurate we can define a reflection coefficient ρ,

$$\rho = \frac{Z_s - Z_d}{Z_s + Z_d} \tag{30}$$

where Z_d is given by (28) and now the gain can be written as:

$$|G|^2 \approx \frac{(1 + |\rho|^2)^2}{4|\rho|^2}. \tag{31}$$

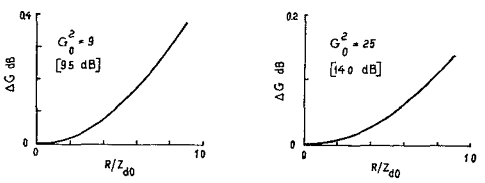

Fig. 7—Maximum deviation from constant gain for constant ρ circle.

Using this approximation it is easy to find the filter giving the desired gain response.

When designing a maximally flat gain response we put a Butterworth restriction on the reflection coefficient given in (30),

$$|\rho|^2 = \frac{\rho_0^2 + \omega^{2n}}{1 + \omega^{2n}}, \tag{32}$$

which inserted in (31) gives

$$|G|^2 \approx \frac{G_0^2 + f(\rho_0)\omega^{2n}}{1 + f(\rho_0)\omega^{2n}}, \tag{33}$$

where

$$\rho_0 = G_0 - \sqrt{G_0^2 - 1}. \tag{34}$$

When designing an equal ripple gain response we put a Chebyshev restriction on the reflection coefficient given in (30),

$$|\rho|^2 = \frac{\rho_0^2 + \epsilon_\rho^2 V_n^2(\omega)}{1 + \epsilon_\rho^2 V_n^2(\omega)} \tag{35}$$

which inserted in (31) gives

$$|G|^2 \approx \frac{G_0^2 + \epsilon_g^2 V_n^2(\omega)}{1 + \epsilon_g^2 V_n^2(\omega)} \tag{36}$$

where

$$\begin{cases} \rho_0 = G_0 - \sqrt{G_0^2 - 1} \\ \epsilon_\rho^2 = \dfrac{(G_0 - \sqrt{G_0^2 - 1})\epsilon_g^2 + G_0 - \sqrt{G_0^2 + \epsilon_g^2}}{\sqrt{G_0^2 + \epsilon_g^2} - \sqrt{G_0^2 - 1}}. \end{cases} \tag{37}$$

Here the fluctuation in Z_d introduced by the Chebyshev ripple is neglected, which is permissible for gains and ripples met in practice.

We now assume that Z_{d0} is constant over the band of interest ($\omega_s\omega_i \approx \omega_{s0}^2$). If the varactor is series tuned, so that the series inductance can be conveniently incorporated in the first series resonator, it is possible to design the filter for a desired gain response by taking the proper low-pass filter with a response given in equation (32) or (35) and transform it into its band-pass equivalent.

B. Nondegenerate Amplifier

When designing filters for the nondegenerate amplifier where $K^2 \neq 1$, the design formulas differ from the degenerate case. For the nondegenerate case the voltage gain G can be written

$$G = \frac{(Z_{d0}/K)^2 + (Z_s^* - R)(Z_s + R/K^2)}{(Z_{d0}/K)^2 - (Z_s + R)(Z_s + R/K^2)}$$

$$= \frac{(KZ_{d0})^2 + (Z_i - K^2R)(Z_i^* + R)}{(KZ_{d0})^2 - (Z_i^* + K^2R)(Z_i^* + R)}. \quad (38)$$

As for the degenerate case we start with an investigation of the Z_s and Z_i^* curves that give a constant voltage gain G_0. These curves intersect the real R axis at two points,

$$\begin{cases} R_{s1} = \dfrac{1}{K}(A - B) \\ R_{s2} = \dfrac{1}{K}(A + B) \end{cases} \qquad \begin{cases} R_{i1} = K(A - B) \\ R_{i2} = K(A + B) \end{cases} \quad (39)$$

where

$$\begin{cases} A - B = \sqrt{\dfrac{G_0 - 1}{G_0 + 1}}\left\{\sqrt{Z_{d0}^2 + \dfrac{R^2}{4}\left[K\sqrt{\dfrac{G_0 - 1}{G_0 + 1}} - \dfrac{1}{K}\sqrt{\dfrac{G_0 + 1}{G_0 - 1}}\right]^2} - \dfrac{R}{2}\left[K\sqrt{\dfrac{G_0 - 1}{G_0 + 1}} + \dfrac{1}{K}\sqrt{\dfrac{G_0 + 1}{G_0 - 1}}\right]\right\} \\ A + B = \sqrt{\dfrac{G_0 + 1}{G_0 - 1}}\left\{\sqrt{Z_{d0}^2 + \dfrac{R^2}{4}\left[K\sqrt{\dfrac{G_0 + 1}{G_0 - 1}} - \dfrac{1}{K}\sqrt{\dfrac{G_0 - 1}{G_0 + 1}}\right]^2} - \dfrac{R}{2}\left[K\sqrt{\dfrac{G_0 + 1}{G_0 - 1}} + \dfrac{1}{K}\sqrt{\dfrac{G_0 - 1}{G_0 + 1}}\right]\right\}. \end{cases} \quad (40)$$

If a circle is drawn through these intersections we have

$$\begin{cases} Z_s = \dfrac{1}{K}A + \dfrac{1}{K}Be^{j\phi} \\ Z_i^* = KA + KBe^{j\phi}. \end{cases} \quad (41)$$

These curves can be used as approximations of the Z_s and Z_i^* curves giving a constant voltage gain G_0. To investigate the accuracy of the approximation we calculate the gain obtained by using the approximation for different values of G_0 and R/Z_{d0}. We choose the impedance ratio $K = \frac{1}{3}$ and the results are given in Fig. 8 as maximum deviation from constant gain.

The calculations show that if a gain fluctuation of 0.2 db is tolerated the approximation is valid for $R/Z_{d0} < 0.9$. This is not a severe restriction as the varactor is too lossy to give amplification when $R/Z_{d0} > 1.0$.

The approximation constant gain curves now can be written as constant ρ curves,

$$\begin{cases} Z_s = \dfrac{1 + \rho_0^2}{1 - \rho_0^2}\dfrac{Z_d}{K} + \dfrac{2\rho_0}{1 - \rho_0^2}\dfrac{Z_d}{K}e^{j\phi} \\ Z_i^* = \dfrac{1 + \rho_0^2}{1 - \rho_0^2}KZ_d + \dfrac{2\rho_0}{1 - \rho_0^2}KZ_de^{j\phi}, \end{cases} \quad (42)$$

where

$$\begin{cases} \rho_0 = \dfrac{A(G_0)}{B(G_0)} - \sqrt{\left[\dfrac{A(G_0)}{B(G_0)}\right]^2 - 1} \\ Z_d = \sqrt{[A(G_0)]^2 - [B(G_0)]^2}. \end{cases} \quad (43)$$

To be able to use the approximations we define a reflection coefficient

$$\rho = \frac{Z_s - Z_d/K}{Z_s + Z_d/K} = \frac{Z_i^* - KZ_d}{Z_i^* + KZ_d} \quad (44)$$

where Z_d is given by (43).

On this reflection coefficient we can impose Butterworth or Chebyshev restrictions to obtain desired gain responses. However, there is no longer any simple relation between the reflection coefficient ρ_0, and the gain G_0 as for the degenerate case. For the previously treated case with $K = \frac{1}{3}$ we plot the reflection coefficient ρ_0 as a function of R/Z_{d0} in Fig. 9.

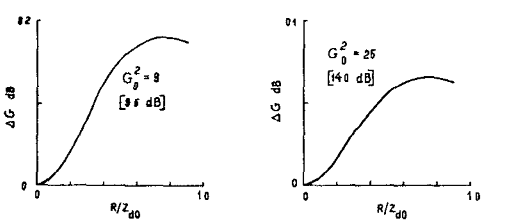

Fig. 8—Maximum deviation from constant gain.

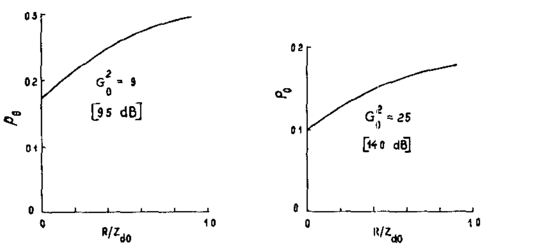

Fig. 9—Reflection coefficient ρ_0 as a function of R/Z_{d0}.

When designing a maximally flat gain response

$$|G|^2 \approx \frac{G_0^2 + f(\rho_0)\omega^{2n}}{1 + f(\rho_0)\omega^{2n}}, \tag{45}$$

we put a Butterworth restriction on the reflection coefficient in (44),

$$|\rho|^2 = \frac{\rho_0^2 + \omega^{2n}}{1 + \omega^{2n}}, \tag{46}$$

where ρ_0 is given in (43).

When designing an equal ripple gain response

$$|G|^2 \approx \frac{G_0^2 + \epsilon_g^2 V_n^2(\omega)}{1 + \epsilon_g^2 V_n^2(\omega)}, \tag{47}$$

we put Chebyshev restriction on the reflection coefficient in (44)

$$|\rho|^2 = \frac{\rho_0^2 + \epsilon_p^2 V_n^2(\omega)}{1 + \epsilon_p^2 V_n^2(\omega)} \tag{48}$$

where ρ_0 is given in (43). The ripple ϵ_p^2 can be determined by calculating the reflection coefficient ρ_1 for

$$|G|^2 = \frac{G_0^2 + \epsilon_g^2}{1 + \epsilon_g^2}$$

from (43) and ϵ_p^2 is given by

$$\rho_1^2 = \frac{\rho_0^2 + \epsilon_p^2}{1 + \epsilon_p^2}. \qquad (49$$

When giving formulas for the equal ripple gain response, we have neglected the fluctuations in Z_d introduced by the Chebyshev ripple, which is permitted for gains and ripples met in practice.

We now assume that Z_d is constant over the band of interest ($\omega_s\omega_i \approx \omega_{s0}\omega_{i0}$). If the varactor is series tuned, so that the series inductance can be conveniently incorporated in the first series resonator, it is possible to design the signal and idler filters for a designed gain response by taking the proper low-pass filters with a response given in (46) or (48), and transform them into their band-pass equivalents. When using this procedure it is assumed, as for the ideal varactor, that the signal and idler circuits are independent of each other. In practical design this assumption does not necessarily hold. This can be compensated by calculating the reactance from the signal circuit at idler frequency, and vice versa, incorporating these reactances in the first series or parallel resonator, according to the actual set up. Then the first series or parallel resonators can be modified by step-by-step approximations.

C. Choice of Initial Varactor Tuning

It has previously been shown that an easy design method can be used if the signal and idler filters start with a series tuning of the varactor, since transformed low-pass filters can then be used.

If the filters are started with a parallel tuning of the varactor, the varactor end of the filters, which consist of the coupling impedance in series with the varactor capacitance and inductance, must be transformed into a parallel circuit. The resistive part of this circuit is frequency dependent, and the proper filters must be obtained by solving (10). This leads to filters of the type given in Fig. 3, with added compensating reactances to fulfill $Z_s = K^2 Z_i^*$. The frequency dependence in the resistive part can be neglected and transformed low-pass circuits can be used, if the reactance from the varactor inductance is small compared to the reactance from the varactor capacitance, but this is ordinarily not the case at microwave frequencies.

VI. Choice of Impedance Ratio

To make full use of the bandwidth potentialities in a varactor, the impedance ratio K should be chosen to give the signal and idler filters the same bandwidth capabilities. The optimum impedance ratio can be determined by applying Bode's[8] theorem on reflection coefficient limitation and its dual formulated by Fano.[9] The formulations are given in Fig. 10.

These theorems hold for low-pass and band-pass circuits alike. To make the signal and idler filters have the same bandwidth capabilities we have: if the varactor is parallel tuned the varactor ends of the signal and idler filters should have the same RC product, and if the varactor is series tuned the varactor ends of the signal and idler filters should have the same R/L ratio. This gives us the following values of K for different tuning of the varactor:

1) for a parallel-tuned ideal varactor,

$$K^2 = 1 \tag{50}$$

2) for a series-tuned ideal varactor,

$$K^2 = \frac{\omega_{s0}^2}{\omega_{i0}^2}. \tag{51}$$

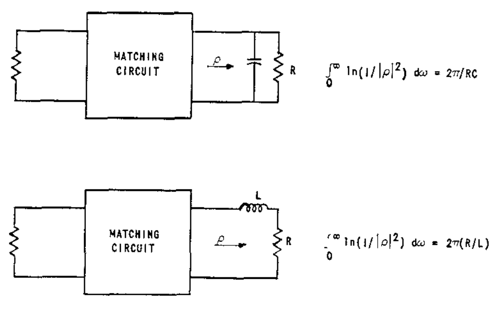

Fig. 10—Theorems on reflection coefficient limitation.

[8] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., Inc., New York, N. Y., pp. 360–367; 1945.

[9] R. M. Fano, "Theoretical limitations on the broadband matching of arbitrary impedances," *J. Franklin Inst.*, vol. 249, pp. 57–83; January, 1950; pp. 139–154, February, 1950.

For a practical varactor we only consider series tuning of the varactor. Reactive elements are added in series to the varactor to obtain series resonances to ω_{s0} and ω_{i0}. We assume that the varactor can be tuned independently at ω_{s0} and ω_{i0}. For three different cases we have:

1) The self-resonant frequency of the varactor is above ω_{s0} and ω_{i0}

$$K^2 = \frac{\omega_{s0}^2}{\omega_{i0}^2}, \tag{52}$$

2) The self-resonant frequency of the varactor is between ω_{s0} and ω_{i0}

$$K^2 = \omega_{s0}^2(1 - \alpha^2)C_dL_s, \tag{53}$$

3) The self-resonant frequency of the varactor is below ω_{s0} and ω_{i0}

$$K^2 = 1. \tag{54}$$

As pointed out by Matthei,[2] it is difficult to tune the varactor independently at ω_{s0} and ω_{i0}, unless there is a first series resonator common to signal and idler. The resonator is series resonant at both ω_{s0} and ω_{i0}, and the value of K can be expressed in the reactance shapes of the resonator,

x_s' = reactance slope at ω_{s0}
x_i' = reactance slope at ω_{i0}

$$K^2 = \frac{x_i'}{x_s'}. \tag{55}$$

VII. Bandwidth Limitations

When designing an amplifier for a certain gain G_0 and increasing the order of the associated Butterworth or Chebyshev filters without limit, the bandwidth goes asymptotically toward a limit $(\Delta\omega)_\infty$. This limit can be determined by applying the theorems on reflection coefficient limitation. With the chosen gain G_0 is associated a reflection coefficient ρ_0 in the signal and idler circuit. In the general case ρ_0 is determined by equation (43). We give the limiting bandwidth for a series-tuned practical varactor in the nondegenerate case. The theorem on reflection coefficient limitation gives

$$(\Delta\omega)_\infty \ln \frac{1}{\rho_0} = \pi \frac{R}{L}. \tag{56}$$

We give $(\Delta\omega)_\infty$ and lim $(\Delta\omega)_\infty$ when $R \to 0$ for the cases where the optimum impedance ratio was determined.

1) the self-resonant frequency is above ω_{s0} and ω_{i0}

$$\left\{\begin{aligned} (\Delta\omega)_\infty &= \frac{\pi}{|\ln \rho_0|}\,\omega_{s0}\omega_{i0}(1-\alpha^2)C_dZ_d \\ \lim_{R\to 0}(\Delta\omega)_\infty &= \frac{\pi}{|\ln(G_0 - \sqrt{G_0^2-1})|}\,\alpha\sqrt{\omega_{s0}\omega_{i0}} \end{aligned}\right. \tag{57}$$

2) the self-resonant frequency is between ω_{s0} and ω_{i0}

$$\left\{\begin{aligned} (\Delta\omega)_\infty &= \frac{\pi}{|\ln \rho_0|}\,\omega_{s0}\sqrt{\frac{(1-\alpha^2)C_d}{L_s}}\,Z_d \\ \lim_{R\to 0}(\Delta\omega)_\infty &= \frac{\pi}{|\ln(G_0 - \sqrt{G_0^2-1})|} \cdot \sqrt{\frac{\omega_{s0}}{\omega_{i0}}}\,\frac{\alpha}{\sqrt{(1-\alpha^2)C_dL_s}}, \end{aligned}\right. \tag{58}$$

3) The self-resonant frequency is above ω_{s0} and ω_{i0}

$$\left\{\begin{aligned} (\Delta\omega)_\infty &= \frac{\pi}{|\ln \rho_0|}\,\frac{Z_d}{L_s} \\ \lim_{R\to 0}(\Delta\omega)_\infty &= \frac{\pi}{|\ln(G_0 - \sqrt{G_0^2-1})|} \cdot \frac{1}{\sqrt{\omega_{s0}\omega_{i0}}}\,\frac{\alpha}{(1-\alpha^2)C_dL_s}; \end{aligned}\right. \tag{59}$$

4) with a common series-resonator

$$\begin{aligned} (\Delta\omega)_\infty &= \frac{\pi}{|\ln \rho_0|}\,\frac{Z_d}{\sqrt{x_s'x_i'}} \\ \lim_{R\to 0}(\Delta\omega)_\infty &= \frac{\pi}{|\ln(G_0 - \sqrt{G_0^2-1})|} \cdot \frac{1}{\sqrt{\omega_{s0}\omega_{i0}}}\,\frac{1}{\sqrt{x_s'x_i'}}\,\frac{\alpha}{(1-\alpha^2)C_d}. \end{aligned} \tag{60}$$

Z_d is determined from equation (43).

These formulas for the limiting bandwidth can be used to determine the optimum ratio between idler and signal frequency, and to determine how the bandwidth is effected by the varactor loss resistance. The ratio

$$\frac{|\ln(G_0 - \sqrt{G_0^2-1})|}{|\ln \rho_0|}\,\frac{Z_d}{Z_{d0}}$$

determines to what degree the loss resistance decreases the limiting bandwidth as compared to the limiting bandwidth of a corresponding lossless varactor. Formulas for the limiting bandwidth for an ideal varactor have previously been given by Kuh and Fukada,[3] and Aron.[10]

Aron[10] also gives the relation between the 3-db bandwidth $(\Delta\omega)_N$ of the reflection coefficient in a Butterworth filter of Nth order and the limiting bandwidth $(\Delta\omega)_\infty$ which can be written

$$\frac{(\Delta\omega)_N}{(\Delta\omega)_\infty} = \frac{2}{\pi}\,\frac{\rho_0^{1/N}}{1-\rho_0^{1/N}}\,|\ln \rho_0|\,\sin\left[\frac{\pi}{2N}(1 - 2\rho_0^2)^{-1/2N}\right]. \tag{61}$$

[10] R. Aron, "Gain bandwidth relations in negative resistance amplifiers," Proc. IRE, vol. 49, pp. 355–356; January, 1961.

VII. Wide Band Design for Lossy Varactors

Calculations have shown that the approximations made in section V are very accurate for most practical wide-band design and that the accuracy increases with increasing gain and decreasing impedance ratio K. When designing amplifiers with low gain below 10 db the made approximations can be less accurate for lossy varactors with large R/Z_{d0} and a different procedure must be used. We restrict ourselves to finding the design formulas for a Butterworth gain response. The procedure used is to start from an expression for the gain (27) or (38) form $|G|^2$, and expand Z_s or Z_i in a Taylor series, as in (9). The Taylor expansion is inserted in the expression for $|G|^2$, and the derivative relations for Butterworth gain can be determined. We treat the nondegenerate case which also includes the degenerate case and assume that Z_{d0} is constant over the band of interest. As Z_{d0} is constant, only even derivatives of R_s and R_i and odd derivatives of X_s and X_i shall be existent. This means that band-pass filters with all series and parallel resonances tuned to ω_{s0} and ω_{i0}, fulfill the relations. The relations between the derivatives giving a Butterworth gain response can be written

$$
\begin{cases}
n=1 \begin{cases} G_0^2\left[(Z_{d0}/K)^2 - (R_s^{(0)}+R)(R_s^{(0)}+R/K^2)\right]^2 = \left[(Z_{d0}/K)^2 + (R_s^{(0)}-R)(R_s^{(0)}+R/K^2)\right]^2 \\ R_s^{(1)} = X_s^{(0)} = 0 \end{cases} \\
n=2 \begin{cases} [X_s^{(1)}]^2 = R_s^{(2)} \dfrac{G_0^2(2R_s^{(0)}+R+R/K^2)\left[(Z_{d0}/K)^2 - (R_s^{(0)}+R)(R_s^{(0)}+R/K^2)\right] + (2R_s^{(0)}-R+R/K^2)\left[(Z_{d0}/K)^2 + (R_s^{(0)}-R)(R_s^{(0)}+R/K^2)\right]}{G_0^2(2R_s^{(0)}+R+R/K^2) - (R/K^2+R)} \\ R_s^{(3)} = X_s^{(2)} = 0 \end{cases}
\end{cases}
\qquad (62)
$$

When the signal filter is determined the idler filter can be determined by scaling, since $Z_s = K^2 Z_i^*$.

By knowing how the Z_s and Z_i^* curves for constant gain are affected by increasing R/Z_{d0}, some rules for selecting proper filters can be made by studying the constant gain curves, as shown in Fig. 11. Increasing R/Z_{d0} makes the Z_s curve for constant gain more ellipse shaped. By studying the Z_s curve in Fig. 11 with associated Butterworth circle, we can make the following rules of thumb for choosing proper filters:

Series-tuned varactor—when choosing a standard filter, the impedance level at the varactor end should be lower than that indicated by the intersections the Z_s curve makes with the real axis;

Parallel-tuned varactor—when choosing a standard filter, the impedance level at the varactor end should be higher than that indicated by the intersections the Z_s curve makes with the real axis.

The impedance level at the varactor end can be determined approximately, by comparing the third equation in (62) with the corresponding equation (10) for a Butterworth filter.

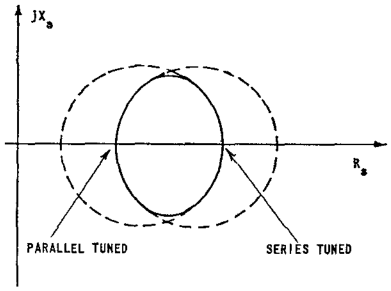

Fig. 11—Z_s curve for constant gain with approximative Butterworth circles.

IX. Wide-Band Design for Varactors with Large Capacitance Swing

For varactors with large capacitance swing (large α), the bandwidth is increased, so that the approximation that Z_{d0} is constant over the band, is no longer adequate. Filters designed for constant Z_{d0} will give a gain response with a slope especially noticable for large idler to signal frequency ratios. If R/Z_{d0} is small enough to allow the previously made approximations, the signal and idler filters shall be designed to match the signal and idler terminations, maximally flat into the frequency dependent Z_d/K and KZ_d for a maximally flat gain response. The design criteria will be to fulfill (10) at the signal and idler terminations and $Z_s = K^2 Z_i^*$ at the varactor end of the filters. For the nondegenerate case a complication is that both Z_d and ρ_0 vary with Z_{d0}. However, a simple calculation shows that ρ_0 varies considerably less than Z_d over the band (For $G_0^2 = 10$, $K = \omega_{s0}/\omega_{i0} = \frac{1}{3}$ $R/Z_{d0}(\omega_{s0}) = 0.2$, Z_d varies ± 35 per cent and $\rho_0 \pm 5$ per cent over an octave-signal bandwidth). Thus it is a reasonable approximation to design for maximally flat matching.

For the degenerate case, only even derivatives of Z_{d0} are existent and a band-pass filter with all series and parallel resonances tuned to ω_{s0} can be used. For the nondegenerate case, the filters will be of the type given in Fig. 3 with a compensating reactance added.

An approximative design method is indicated by Matthei.[2] The filters are designed for constant Z_{d0} and then the resonators are detuned by cut and try to give an acceptably smooth gain curve.

X. Stability

When designing wide-band parametric amplifiers, stability must be considered. The criterion for stability is that the voltage gain G has no poles in the right half frequency plane (Re $[p]<0$).[11] Using the gain expressions in (17), (27) and (38) we have the following stability criteria:

1) for an ideal varactor,

$$\begin{cases} \frac{1}{K} Z_{d0}(p) - Z_s(p) = 0 \\ K Z_{d0}(p) - Z_i^*(p) = 0 \end{cases} \tag{63}$$

has no solution for $Re[p]<0$;

2) for a practical varactor degenerate case,

$$[Z_{d0} - R] - Z_s = 0 \tag{64}$$

has no solution for Re $[p]<0$;

3) For a practical varactor nondegenerate case,

$$\begin{cases} \left[\sqrt{\left(\frac{Z_{d0}}{K}\right)^2 + \frac{R^2}{4}\left(\frac{1}{K^2} - 1\right)^2} - \frac{R}{2}\left(\frac{1}{K^2} + 1\right)\right] - Z_s = 0 \\ \left[\sqrt{(KZ_{d0})^2 + \frac{R^2}{4}(1-K^2)^2} - \frac{R}{2}(1+K^2)\right] - Z_i^* = 0 \end{cases} \tag{65}$$

has no solutions for Re $[p]<0$.

Applying the Nyquist criterion as formulated by Hughes,[12] we can reformulate the stability criteria. We take the most general case of a practical varactor in the nondegenerate case.

To have a stable amplifier, a complex plot of $Z_s(\omega_s)$ shall not encircle $(1/K)Z_{\text{crit}}$, and a complex plot of $Z_i^*(\omega_i)$ shall not encircle KZ_{crit}

$$Z_{\text{crit}} = \sqrt{Z_{d0}^2 + \frac{R^2}{4}\left(\frac{1}{K} - K\right)^2} - \frac{R}{2}\left(\frac{1}{K} + K\right). \tag{66}$$

This means that for a series-tuned varactor, the impedance level is higher at the terminations than at the varactor end, and for a parallel-tuned varactor the impedance level is lower at the terminations than at the varactor end. In Fig. 12, we show the Z_s plot of two possible Butterworth filters of second order.

As pointed out by Aron,[10] stability is no limitation, since of possible Butterworth of Chebyshev filters, the filters with maximum bandwidth have the poles of ρ in the left half plane (Re $[p]<0$), which ensures stability. This means that when determining proper low-pass filters with Butterworth or Chebyshev response, the solutions with the smallest L:s and C:s should be chosen.

[11] It can be shown that the signal-idler interaction is valid in the complex frequency plane if signal and idler voltages of the form $e^{\gamma t}\cos wt$ are introduced into (11).

[12] W. L. Hughes, "Nonlinear Electrical Networks," Ronald Co., New York, N. Y., pp. 166–168; 1960.

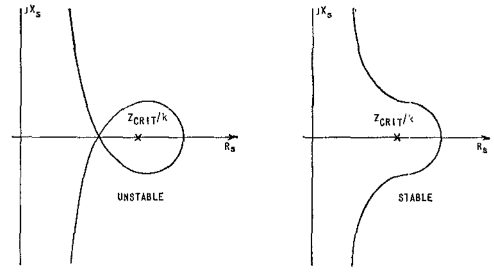

Fig. 12—Z_s plot for two Butterworth filters of second order.

XI. Design Examples

We start with determining the same low-pass filters that will give stable parametric amplifiers. In Fig. 13 stable Butterworth filters of second and third order are given.

We will now apply the described design method to the design of a degenerate parametric amplifier with a single sideband gain of 12 db. The design will be made for maximally flat gain responses of first, second and third order. The diode used has a pumping factor $\alpha=0.25$ and Z_{d0} and ω_{s0} are normalized to unity. The diode loss resistance is $R/Z_{d0}=0.1$. The diode series resonance is above the operating frequency.

The design procedure is as follow. Z_d and ρ_0 are determined from (28) and (29) ($G_0^2=16$, $\rho_0=0.127$, $Z_d=0.897$). (Elsewhere,[13] the author has given diagrams of ρ_0, Z_d and maximum gain deviation for different gains, impedance ratios and varactor losses. These diagrams are helpful in numerical design.) The inductance needed to resonate the varactor is determined ($L=4$). Using the formulas in Fig. 13, the low-pass network is determined ($R_1=Z_d$, $L_1=L$). The band-pass filter is determined by resonating the low-pass network to ω_{s0}. The gain is determined by (27). The result is given in Fig. 14.

We also give a simple illustration of how to compensate for frequency dependence in Z_{d0} as discussed in section IX. We assume that we have an ideal varactor with $K=\omega_{s0}/\omega_{i0}=\frac{1}{5}$ and $\alpha=0.5$. The circuits used are given in Fig. 15. The varactor is series tuned. ω_{s0} is normalized to unity.

The compensated circuit is dimensional to fulfill (10) at the signal and idler terminations and to make $X_s(\omega_{s0})=X_i(\omega_{i0})=0$. The midband gain is chosen to 14 db. The gain for the two circuits are shown in Fig. 16.

The method gives no great advantage for the simple circuit treated, but can be extended to maximally-flat gain responses of higher order, although this leads to complicated equations.

[13] B. T. Henoch, "A New Method for Designing Wide-Band Parametric Amplifiers," Stanford Electronics Labs., Stanford University, Stanford, Calif., Tech. Rept. No. 213-1; December, 1962.

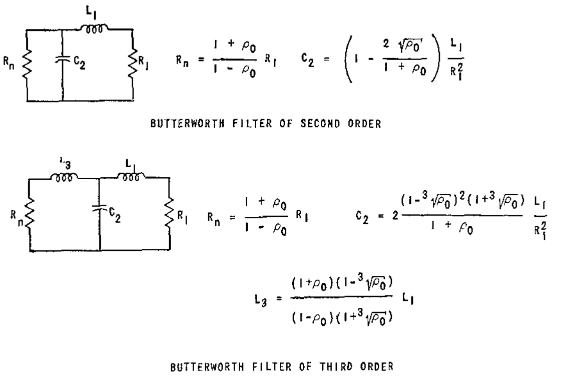

Fig. 13—Stable low-pass filters with Butterworth response of second and third order.

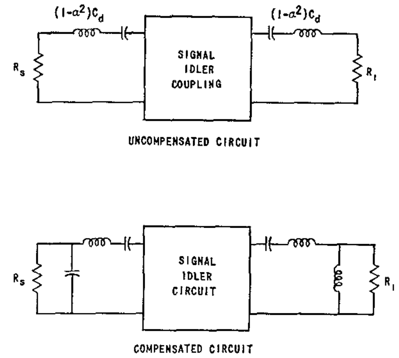

Fig. 15—Circuits for nondegenerate parametric amplifier.

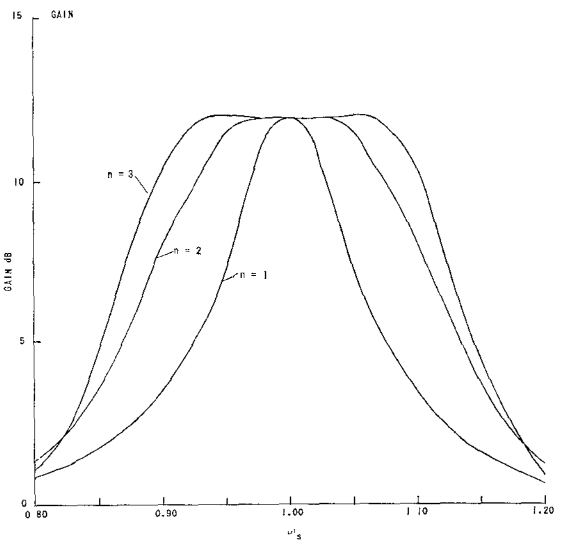

Fig. 14—Single sideband gain for a degenerate parametric amplifier with maximally-flat gain response of first, second and third order.

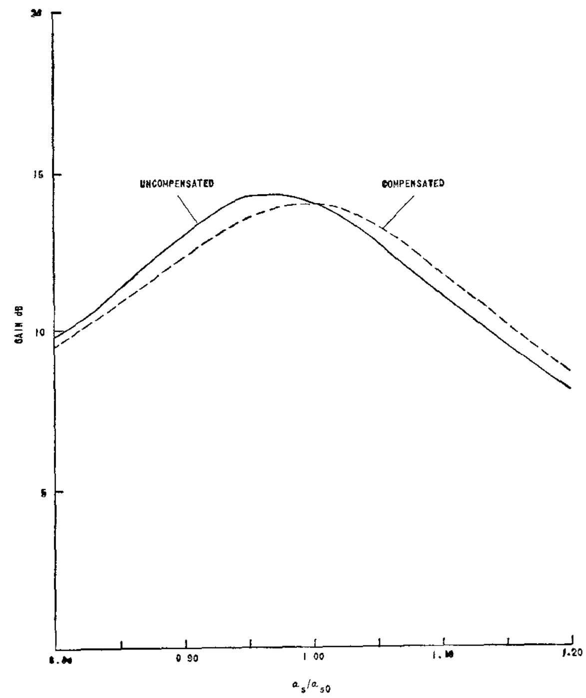

Fig. 16—Gain for a series-tuned nondegenerate amplifier with and without compensation for frequency dependance in Z_{d0}.

XIII. Conclusions

A simple method for designing wide-band parametric amplifiers has been described. The method is applicable to most practical varactors if the varactor is initially series tuned.

The problems remaining to be solved are: filter design for an initially parallel-tuned varactor, and filter design compensating for the frequency dependence in the coupling impedance. Both problems are associated with the problem of designing filters that match a constant source impedance, maximally flat into a frequency dependent resistance.

Acknowledgment

The author wishes to thank Prof. H. Heffner, Y. Kvaerna and Prof. R. Newcomb for helpful discussions and comments on the manuscript.

The author also greatefully acknowledges the support of a scholarship from the Delegation of Solid State Physics, Stockholm, Sweden.